

KOBE-TH-00-05  
hep-th/0008158

## On the gauge parameter dependence of QED

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November 2000

PACS numbers: 11.10.Gh, 11.15.-q

Keywords: renormalization, gauge field theories

### Abstract

The gauge parameter dependence of QED in the covariant gauge was given explicitly long time ago by Landau and Khalatnikov. We elucidate their result by giving two new derivations. The first derivation uses the BRST invariance of the theory with a Stückelberg field, which is a non-interacting fictitious Goldstone boson field. The second derivation is more straightforward but calculational.

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The purpose of this paper is to introduce a very simple trick for determining the dependence of abelian gauge theories on the gauge fixing parameter. It is true that important physics lies only in the gauge independent part of the theory, and anything dependent on the gauge fixing parameter is unphysical. Nevertheless, perturbation theory of the manifestly renormalizable gauge theories cannot be formulated in a gauge invariant way, and it is important to have a total control over the gauge dependence of the correlation functions.

The gauge dependence of QED was discussed long ago by Landau and Khalatnikov [1]. The familiar lagrangian of QED with electrons is given by

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial_\mu A_\mu)^2 + \bar{\psi}\left(\frac{1}{i}\not{\partial} - e\not{A} + iM\right)\psi \quad (1)$$

where  $\xi$  is a gauge fixing parameter.<sup>1</sup> Landau and Khalatnikov stated that the correlation functions in two different gauges  $\xi, \xi'$  are related by<sup>2</sup>

$$\langle A_{\mu_1} \dots \psi \dots \bar{\psi} \dots \rangle_{\xi'} = \langle (A_{\mu_1} + \partial_{\mu_1} \phi) \dots e^{ie\phi} \psi \dots e^{-ie\phi} \bar{\psi} \dots \rangle_{\xi} \quad (2)$$

where  $\phi$  is a free real scalar field whose propagator is given by

$$\langle \phi(x) \phi(0) \rangle \equiv (\xi' - \xi) \int_k \frac{e^{ikx}}{(k^2)^2} \quad (3)$$

This implies physically the free field nature of the longitudinal mode of the photon. Landau and Khalatnikov justified their result by showing its consistency with the Ward identities. The specific cases for the two- and three-point functions were also verified by the method of generating functionals in refs. [2, 3].<sup>3</sup>

The aim of this paper is to rederive Eq. (2) in an illuminating way. By introducing a Stückelberg field, we will derive Eq. (2) as a simple consequence of the BRST invariance. We will first prove the generalization of Eq. (2) for the massive QED. Then, the result for QED with massless photons can be obtained by taking the massless limit.

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<sup>1</sup>We will use the euclidean metric throughout the paper. The action  $S$  is the integral of the lagrangian density  $\mathcal{L}$  over 4-dimensional euclidean space, and the weight of functional integration is given by  $e^{-S}$ .

<sup>2</sup>Actually they gave a result applicable to any gauge fixing function of  $\partial_\mu A_\mu$ .

<sup>3</sup>In ref. [2], only the gauge parameter dependence of the electron propagator was given explicitly although the arguments there can be generalized. In ref. [3] the change of the generating functional under an arbitrary infinitesimal change of the gauge fixing function was given, and Eq. (2), with  $\phi$  contracted, was obtained explicitly for the electron propagator and the vertex function.

Let us consider the following lagrangian for the massive QED:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\varphi - mA_\mu)^2 + \frac{1}{2\xi}(\partial_\mu A_\mu - \xi m\varphi)^2 \\
&\quad + \bar{\psi} \left( \frac{1}{i}\not{\partial} - e\not{A} + iM \right) \psi + \partial_\mu \bar{c} \partial_\mu c + \xi m^2 \bar{c} c \\
&= \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial \cdot A)^2 + \frac{m^2}{2}A_\mu^2 + \frac{1}{2}((\partial_\mu\varphi)^2 + \xi m^2\varphi^2) \\
&\quad + \bar{\psi} \left( \frac{1}{i}\not{\partial} - e\not{A} + iM \right) \psi + \partial_\mu \bar{c} \partial_\mu c + \xi m^2 \bar{c} c
\end{aligned} \tag{4}$$

The real scalar field  $\varphi$ , known as the Stückelberg field, is the Goldstone boson field which gives a mass  $m$  to the photon. The exponentiated field  $e^{i\frac{e}{m}\varphi}$  is a charged scalar with its magnitude frozen. We have chosen the  $R_\xi$  gauge so that  $\varphi$  becomes a free massive field of squared mass  $\xi m^2$ . Ignoring the  $\varphi$  and the anticommuting Faddeev-Popov (FP) ghost fields  $c, \bar{c}$  (or equivalently integrating them out), the lagrangian reduces to the standard lagrangian for the massive QED with electrons in the covariant gauge.

The lagrangian (4) is invariant under the following BRST transformation:

$$\begin{aligned}
\delta_\epsilon A_\mu &= \epsilon \partial_\mu c, & \delta_\epsilon \varphi &= m\epsilon c \\
\delta_\epsilon \psi &= ie\epsilon c\psi, & \delta_\epsilon \bar{\psi} &= -ie\epsilon c\bar{\psi} \\
\delta_\epsilon c &= 0, & \delta_\epsilon \bar{c} &= \epsilon \frac{1}{\xi}(\partial \cdot A - \xi m\varphi)
\end{aligned} \tag{5}$$

where  $\epsilon$  is an arbitrary anticommuting constant. Out of the four fields  $A_\mu, \psi, \bar{\psi}$ , and  $\varphi$  we can construct three gauge and BRST invariant fields:

$$A'_\mu \equiv A_\mu - \frac{1}{m}\partial_\mu\varphi \tag{6}$$

$$\psi' \equiv e^{-i\frac{e}{m}\varphi}\psi, \quad \bar{\psi}' \equiv e^{i\frac{e}{m}\varphi}\bar{\psi} \tag{7}$$

It is easy to compute the  $\xi$  dependence of the lagrangian:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \xi} &= -\frac{1}{2\xi^2}(\partial \cdot A)^2 + \frac{1}{2}m^2\varphi^2 + m^2\bar{c}c \\
&= -\frac{1}{2\xi}\frac{1}{\epsilon}\delta_\epsilon[\bar{c}(\partial \cdot A + \xi m\varphi)] + \frac{1}{2\xi}\bar{c}(-\partial^2 + \xi m^2)c
\end{aligned} \tag{8}$$

This implies that the correlations of BRST invariant fields that are independent of the FP ghosts do not depend on the gauge fixing parameter  $\xi$ .

Therefore, the correlation functions of  $A'_\mu, \psi', \bar{\psi}'$  are independent of  $\xi$ :

$$\langle A'_{\mu_1} \dots \psi' \dots \bar{\psi}' \dots \rangle_{\xi'} = \langle A'_{\mu_1} \dots \psi' \dots \bar{\psi}' \dots \rangle_{\xi} \quad (9)$$

The explicit gauge dependence of the correlation functions can be obtained by contracting the free scalar  $\varphi$  in this formula.

We now wish to derive the Landau-Khalatnikov result (2) for the massive QED:

$$\langle A_{\mu_1} \dots \psi \dots \bar{\psi} \dots \rangle_{\xi'} = \langle (A_{\mu_1} + \partial_{\mu_1} \phi) \dots \psi e^{ie\phi} \dots \bar{\psi} e^{-ie\phi} \dots \rangle_{\xi} \quad (10)$$

where the propagator of the free scalar field  $\phi$  is given by

$$\begin{aligned} \langle \phi(x) \phi(0) \rangle &\equiv \frac{1}{m^2} \int_k e^{ikx} \left( \frac{1}{k^2 + \xi m^2} - \frac{1}{k^2 + \xi' m^2} \right) \\ &= (\xi' - \xi) \int_k e^{ikx} \frac{1}{(k^2 + \xi m^2)(k^2 + \xi' m^2)} \end{aligned} \quad (11)$$

Instead of proving Eq. (10) directly, we prove the following equivalent relation that is obtained by substituting Eq. (10) into Eq. (9):

$$\begin{aligned} &\left\langle \left( A_{\mu_1} + \partial_{\mu_1} \left( \phi - \frac{1}{m} \varphi_{\xi'} \right) \right) \dots \psi e^{ie(\phi - \frac{1}{m} \varphi_{\xi'})} \dots \bar{\psi} e^{-ie(\phi - \frac{1}{m} \varphi_{\xi'})} \dots \right\rangle_{\xi} \\ &= \left\langle \left( A_{\mu_1} - \partial_{\mu_1} \frac{1}{m} \varphi_{\xi} \right) \dots \psi e^{-i\frac{e}{m} \varphi_{\xi}} \dots \bar{\psi} e^{i\frac{e}{m} \varphi_{\xi}} \dots \right\rangle_{\xi} \end{aligned} \quad (12)$$

where  $\varphi_{\xi}$  is the Stückelberg field for the gauge fixing parameter  $\xi$  with the propagator

$$\langle \varphi_{\xi}(x) \varphi_{\xi}(0) \rangle = \Delta(x - y; \xi m^2) \equiv \int_k \frac{e^{ikx}}{k^2 + \xi m^2} \quad (13)$$

Eq. (12) is valid if and only if

$$\left\langle e^{ie(\phi - \frac{1}{m} \varphi_{\xi'})} \dots e^{-ie(\phi - \frac{1}{m} \varphi_{\xi'})} \dots \right\rangle = \left\langle e^{-i\frac{e}{m} \varphi_{\xi}} \dots e^{i\frac{e}{m} \varphi_{\xi}} \dots \right\rangle \quad (14)$$

But this is equivalent to

$$\langle \phi \phi \rangle = \frac{1}{m^2} (\langle \varphi_{\xi} \varphi_{\xi} \rangle - \langle \varphi_{\xi'} \varphi_{\xi'} \rangle), \quad (15)$$

which is precisely the definition of the propagator of  $\phi$  given in Eq. (11). This concludes the proof of Eq. (10).<sup>4</sup> The propagator of  $\phi$  is the difference of the propagators of the Stückelberg fields in two gauges.

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<sup>4</sup>Eq. (12.9.21) of ref. [4] gives the  $\xi$  derivative of arbitrary correlation functions. By integrating the equation over  $\xi$ , Eqs. (10) of the main text can be obtained in principle.

The original result (2) for QED with massless photons can be derived from Eq. (10) simply by taking the limit  $m \rightarrow 0$ ; in the limit we get the massless propagator (3) from the massive propagator (11). However, the proof itself does not go through for  $m = 0$ . With  $m = 0$  the BRST invariant lagrangian (4) still makes sense, but the definitions (6,7) suffer from the diverging factor  $\frac{1}{m} \rightarrow \infty$ . A slight modification is necessary for the proof. We introduce the following lagrangian:

$$\mathcal{L} = \frac{1}{4}F^2 + \frac{1}{2\xi}(\partial \cdot A)^2 - \frac{1}{2\xi}(\partial^2 \varphi)^2 + \bar{\psi} \left( \frac{1}{i} \not{\partial} - e \not{A} + iM \right) \psi + \partial_\mu \bar{c} \partial_\mu c \quad (16)$$

This is invariant under the BRST transformation:

$$\begin{aligned} \delta_\epsilon A_\mu &= \epsilon \partial_\mu c, & \delta_\epsilon \varphi &= \epsilon c \\ \delta_\epsilon \psi &= ie\epsilon c \psi, & \delta_\epsilon \bar{\psi} &= -ie\epsilon c \bar{\psi} \\ \delta_\epsilon c &= 0, & \delta_\epsilon \bar{c} &= \epsilon \frac{1}{\xi} (\partial \cdot A - \partial^2 \varphi) \end{aligned} \quad (17)$$

This transformation is strictly nilpotent. From the BRST invariance of the redefined fields

$$\begin{aligned} A'_\mu &\equiv A_\mu - \partial_\mu \varphi \\ \psi' &\equiv e^{-ie\varphi} \psi, & \bar{\psi}' &\equiv e^{ie\varphi} \bar{\psi}, \end{aligned} \quad (18)$$

the rest of the proof follows exactly the same way.

There is a more straightforward way of deriving the gauge parameter dependence (10). Before proceeding with the derivation, however, let us work out a few concrete consequences of the Landau-Khalatnikov equation (10) for completeness of the paper. We first consider the electron propagator. Eq. (10) implies

$$\langle \psi(x) \bar{\psi}(y) \rangle_\xi = \exp \left[ \frac{\xi e^2}{(4\pi)^2} \left( -\mathcal{D}(0; \xi m^2) + \mathcal{D}(x-y; \xi m^2) \right) \right] \langle \psi(x) \bar{\psi}(y) \rangle_{\xi=0} \quad (19)$$

where

$$\frac{1}{(4\pi)^2} \mathcal{D}(x; \xi m^2) \equiv \frac{1}{\xi m^2} \left( \Delta(x; 0) - \Delta(x; \xi m^2) \right) = \int_k \frac{e^{ikx}}{k^2(k^2 + \xi m^2)} \quad (20)$$

The value of  $\mathcal{D}(0; \xi m^2)$  is ultraviolet (UV) divergent, and in the dimensional regularization it is calculated as

$$\frac{1}{(4\pi)^2} \mathcal{D}(0; \xi m^2) = \mu^\epsilon \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k^2 + \xi m^2)} = \frac{1}{(4\pi)^2} \left( \frac{2}{\epsilon} + 1 - \ln \frac{\xi m^2}{\bar{\mu}^2} \right) \quad (21)$$

where  $D \equiv 4 - \epsilon$ , and  $\bar{\mu}^2 \equiv 4\pi\mu^2 e^{-\gamma}$  ( $\gamma$  is the Euler constant). Let us introduce renormalization constants as follows:

$$\begin{aligned} A_{\mu,r} &\equiv \frac{1}{\sqrt{Z_3}} A_\mu, & \psi_r &\equiv \frac{1}{\sqrt{Z_2(\xi)}} \psi \\ e_r^2 &\equiv Z_3 e^2, & m_r^2 &\equiv Z_3 m^2, & \xi_r &\equiv \frac{1}{Z_3} \xi \end{aligned} \quad (22)$$

where  $Z_3$  is independent of  $\xi$ . Eqs. (19, 20, 21) imply that, in the minimal subtraction (MS) scheme, the wave function renormalization constant depends on the gauge fixing parameter as

$$Z_2(\xi) = \exp \left[ - \frac{\xi_r e_r^2}{(4\pi)^2} \frac{2}{\epsilon} \right] Z_2(0). \quad (23)$$

This relation was first obtained by Johnson and Zumino [5].<sup>5</sup> After renormalization, Eq. (19) gives [1, 2, 3]<sup>6</sup>

$$\begin{aligned} &\left\langle \psi_r(y) \bar{\psi}_r(z) \right\rangle_{\xi_r} / \left\langle \psi_r(y) \bar{\psi}_r(z) \right\rangle_{\xi_r=0} \\ &= \exp \left[ \frac{\xi_r e_r^2}{(4\pi)^2} \left( \ln \frac{\xi_r m_r^2}{\bar{\mu}^2} - 1 + \mathcal{D}(y - z; \xi_r m_r^2) \right) \right] \end{aligned} \quad (24)$$

where  $\mathcal{D}$ , which is UV finite for non-vanishing arguments, is defined by Eq. (20).

We next consider the three-point function. Eq. (10) gives, after renormalization,

$$\begin{aligned} &\left\langle A_{\mu,r}(x) \psi_r(y) \bar{\psi}_r(z) \right\rangle_{\xi_r} = \exp \left[ \frac{\xi_r e_r^2}{(4\pi)^2} \left( \ln \frac{\xi_r m_r^2}{\bar{\mu}^2} - 1 + \mathcal{D}(y - z; \xi_r m_r^2) \right) \right] \\ &\times \left[ \left\langle A_{\mu,r}(x) \psi_r(y) \bar{\psi}_r(z) \right\rangle_0 \right. \\ &\quad \left. + \frac{i \xi_r e_r}{(4\pi)^2} \left( \partial_\mu \mathcal{D}(x - y; \xi_r m_r^2) - \partial_\mu \mathcal{D}(x - z; \xi_r m_r^2) \right) \left\langle \psi_r(y) \bar{\psi}_r(z) \right\rangle_0 \right] \end{aligned} \quad (25)$$

This was obtained in refs. [1, 3] for  $m_r^2 = 0$ .

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<sup>5</sup>A large regulator mass was used as a UV cutoff in [5]. The calculation with the dimensional regularization was first made by Collins [4, 6] and Lautrup [7].

<sup>6</sup>See also ref. [8] for a derivation using the functional method.

For arbitrary correlation functions in the MS scheme, we find

$$\begin{aligned}
& \left\langle A''_{\mu,r} \dots \psi_r(y_1) \dots \psi_r(y_F) \bar{\psi}_r(z_1) \dots \bar{\psi}_r(z_F) \right\rangle_{\xi_r} / \left\langle A_{\mu,r} \dots \psi_r \dots \bar{\psi}_r \dots \right\rangle_0 \\
&= \exp \left[ \frac{\xi_r e_r^2}{(4\pi)^2} \left\{ F \left( \ln \frac{\xi_r m_r^2}{\bar{\mu}^2} - 1 \right) + \sum_{i,j} \mathcal{D}(y_i - z_j; \xi_r m_r^2) \right. \right. \\
&\quad \left. \left. - \sum_{i < j} \left( \mathcal{D}(y_i - y_j; \xi_r m_r^2) + \mathcal{D}(z_i - z_j; \xi_r m_r^2) \right) \right\} \right] \quad (26)
\end{aligned}$$

where only the transverse part

$$A''_\mu \equiv A_\mu - \partial_\mu \frac{1}{\partial^2} \partial \cdot A \quad (27)$$

has been considered for simplicity. Eq. (26) implies that the correlation functions of the elementary fields at large distances are independent of the gauge fixing parameter. This would imply the gauge independence of the S-matrix elements in the Minkowski space.

Now, let us proceed with the second derivation of Eq. (10). We start with the following lagrangian for the massive QED:

$$\begin{aligned}
\mathcal{L} = & \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m^2 A_\mu^2 + \frac{1}{2\xi} (\partial \cdot A)^2 + \bar{\psi} \left( \frac{1}{i} \not{\partial} - e \not{A} + iM \right) \psi \\
& + \frac{1}{2(\xi' - \xi)} \phi (-\partial^2 + \xi m^2) (-\partial^2 + \xi' m^2) \phi \quad (28)
\end{aligned}$$

where we have added a non-interacting scalar field  $\phi$ , which is the same field that appears in Eq. (10). Since  $\phi$  is decoupled, we can integrate it out to get the standard lagrangian for the massive QED with the gauge fixing parameter  $\xi$ . We introduce a change of variables

$$A'_\mu = A_\mu + \partial_\mu \phi, \quad \psi' = e^{ie\phi} \psi, \quad \bar{\psi}' = e^{-ie\phi} \bar{\psi} \quad (29)$$

This is nothing but a gauge transformation with the gauge function  $\phi$ . Note that the right-hand side of Eq. (10) is the correlation of  $A'_\mu, \psi', \bar{\psi}'$  using the above lagrangian (28). In terms of the redefined fields, we can rewrite the lagrangian as

$$\begin{aligned}
\mathcal{L} = & \frac{1}{4} F_{\mu\nu}'^2 + \frac{1}{2} m^2 (A'_\mu - \partial_\mu \phi)^2 + \frac{1}{2\xi} (\partial \cdot A' - \partial^2 \phi)^2 \\
& + \bar{\psi}' \left( \frac{1}{i} \not{\partial} - e \not{A}' + iM \right) \psi' + \frac{1}{2(\xi' - \xi)} \phi (-\partial^2 + \xi m^2) (-\partial^2 + \xi' m^2) \phi
\end{aligned}$$

$$\begin{aligned}
= & \frac{1}{4}F_{\mu\nu}'^2 + \frac{1}{2}m^2 A'^2 + \frac{1}{2\xi}(\partial \cdot A')^2 + \overline{\psi}' \left( \frac{1}{i}\not{\partial} - eA' + iM \right) \psi' \\
& + \frac{1}{\xi}\phi \left( -\partial^2 + \xi m^2 \right) \partial \cdot A' + \frac{\xi'}{2\xi(\xi' - \xi)}\phi \left( -\partial^2 + \xi m^2 \right)^2 \phi
\end{aligned} \tag{30}$$

Integrating out  $\phi$ , we get

$$\mathcal{L}' = \frac{1}{4}F_{\mu\nu}'^2 + \frac{1}{2}m^2 A'^2 + \frac{1}{2\xi'}(\partial \cdot A')^2 + \overline{\psi}' \left( \frac{1}{i}\not{\partial} - eA' + iM \right) \psi', \tag{31}$$

which is a lagrangian with a new gauge fixing parameter  $\xi'$ . This implies that the correlation of  $A'_\mu, \psi', \overline{\psi}'$  using the lagrangian (28) is the same as the correlation of the same fields using the lagrangian (31). Hence, we obtain Eq. (10).

We wish to apply the techniques developed above to the gauge dependence of the abelian Higgs theory both in the covariant gauge and in the  $R_\xi$  gauge. In the covariant gauge the theory is defined by the lagrangian

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial \cdot A)^2 + |(\partial_\mu - ieA_\mu)\phi|^2 + M^2|\phi|^2 + \frac{\lambda}{4}|\phi|^4 \tag{32}$$

The  $\xi$  dependence of the correlation functions of  $A_\mu, \phi, \phi^*$  can be obtained in the same way as for the QED with electrons. Let us consider the implications of the  $\xi$  dependence thus obtained. In the Higgs phase the correlation function

$$\langle A_\mu \dots \phi(y_1) \dots \phi(y_B) \phi^*(z_1) \dots \phi^*(z_{\bar{B}}) \rangle \tag{33}$$

is non-vanishing even if  $B \neq \bar{B}$ . The correlation functions for  $B \neq \bar{B}$  have  $\xi$  dependent infrared (IR) divergences. For example, we find, after UV renormalization, that

$$\langle \phi_r \rangle_{\xi_r} = e^{\frac{\xi_r e_r^2}{(4\pi)^2} \frac{1}{\epsilon}} \langle \phi_r \rangle_0 \tag{34}$$

For the two-point functions, we obtain

$$\begin{aligned}
2 \langle \Re \phi_r(x) \Re \phi_r(y) \rangle_{\xi_r} &= \left( \pi e^\gamma \mu^2 (x-y)^2 \right)^{-\frac{\xi_r e_r^2}{(4\pi)^2}} \langle \phi(x) \phi^*(y) \rangle_0 \\
&+ e^{\frac{\xi_r e_r^2}{(4\pi)^2} \frac{4}{\epsilon}} \left( \pi e^\gamma \mu^2 (x-y)^2 \right)^{\frac{\xi_r e_r^2}{(4\pi)^2}} \langle \phi(x) \phi(y) \rangle_0
\end{aligned} \tag{35}$$

$$\begin{aligned}
2 \langle \Im \phi_r(x) \Im \phi_r(y) \rangle_{\xi_r} &= \left( \pi e^\gamma \mu^2 (x-y)^2 \right)^{-\frac{\xi_r e_r^2}{(4\pi)^2}} \langle \phi(x) \phi^*(y) \rangle_0 \\
&- e^{\frac{\xi_r e_r^2}{(4\pi)^2} \frac{4}{\epsilon}} \left( \pi e^\gamma \mu^2 (x-y)^2 \right)^{\frac{\xi_r e_r^2}{(4\pi)^2}} \langle \phi(x) \phi(y) \rangle_0
\end{aligned} \tag{36}$$



The second terms on the right-hand sides are IR divergent for non-vanishing  $\xi_r$ . All IR divergences in the covariant gauge depend on  $\xi_r$ , and they can be determined explicitly.

Finally, we consider the Higgs theory in the  $R_\xi$  gauge [9]. In this case the fictitious Goldstone boson does not decouple for the massive photon, and we cannot obtain the  $\xi$  dependence of the correlations of elementary fields. The lagrangian in the  $R_\xi$  gauge is given by

$$\begin{aligned} \mathcal{L}_{R_\xi} = & \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial \cdot A - \xi ev\chi)^2 \\ & + |(\partial_\mu - ieA_\mu)\phi|^2 + M^2|\phi|^2 + \frac{\lambda}{4}|\phi|^4 + \partial_\mu \bar{c}\partial_\mu c + \xi e^2 v \rho \bar{c}c \end{aligned} \quad (37)$$

where  $\rho, \chi$  are the real and imaginary parts of  $\phi$ :

$$\phi = \frac{1}{\sqrt{2}}(\rho + i\chi) \quad (38)$$

The mass parameter  $v$  has been introduced to remove the tree-level mixing between  $\chi$  and  $\partial \cdot A$ . A Stückelberg field can be introduced as

$$\begin{aligned} \mathcal{L} = & \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu \varphi - mA_\mu)^2 + \frac{1}{2\xi}(\partial \cdot A - \xi ev\chi - \xi m\varphi)^2 \\ & + |(\partial_\mu - ieA_\mu)\phi|^2 + M^2|\phi|^2 + \frac{\lambda}{4}|\phi|^4 \\ & + \partial_\mu \bar{c}\partial_\mu c + \xi(e^2 v \rho + m^2)\bar{c}c \end{aligned} \quad (39)$$

The  $\varphi$  field is not free anymore; it couples to  $\chi$  through the term  $\xi ev m \varphi \chi$ . Regarding the field  $\chi$  as a source,  $\varphi$  can be integrated out. In the massless limit  $m \rightarrow 0$ , the field  $\varphi$  decouples from  $\chi$ , and the lagrangian (39) reduces to (37). But, as we will see shortly, the effect of the coupling of  $\varphi$  with  $\chi$  remains even in the massless limit.

The lagrangian (39) is invariant under the following BRST transformation:

$$\begin{aligned} \delta_\epsilon A_\mu &= \epsilon \partial_\mu c, & \delta_\epsilon \varphi &= m \epsilon c \\ \delta_\epsilon \rho &= -\epsilon \epsilon c \chi, & \delta_\epsilon \chi &= \epsilon \epsilon c \rho \\ \delta_\epsilon c &= 0, & \delta_\epsilon \bar{c} &= \epsilon \frac{1}{\xi}(\partial \cdot A - \xi ev\chi - \xi m\varphi) \end{aligned} \quad (40)$$

and the BRST invariant fields are defined by

$$A''_\mu \equiv A_\mu - \partial_\mu \frac{1}{\partial^2} \partial \cdot A \quad (41)$$

$$\phi' \equiv e^{-i\frac{e}{m}\varphi}\phi, \quad \phi'^* \equiv e^{i\frac{e}{m}\varphi}\phi^* \quad (42)$$

Using the  $\xi$  independence of the correlation of  $A''_\mu, \phi', \phi'^*$ , we obtain

$$\begin{aligned}
& \left\langle A''_\mu \dots \phi(y_1) \dots \phi(y_B) \phi^*(z_1) \dots \phi^*(z_{\bar{B}}) \right. \\
& \quad \times \exp \left[ \frac{1}{2} \xi^2 e^2 v^2 m^2 \int_{r, r'} \chi(r) \Delta(r - r'; \xi m^2) \chi(r') \right. \\
& \quad \left. \left. + i \xi e^2 v \int_r \chi(r) \left( \sum_{1 \leq i \leq B} \Delta(r - y_i; \xi m^2) - \sum_{1 \leq i \leq \bar{B}} \Delta(r - z_i; \xi m^2) \right) \right] \right\rangle_\xi \\
& = \exp \left[ \frac{\xi e^2}{(4\pi)^2} \left( -\frac{1}{2} (B + \bar{B}) \mathcal{D}(0; \xi m^2) + \sum_{i \leq B, j \leq \bar{B}} \mathcal{D}(y_i - z_j; \xi m^2) \right. \right. \\
& \quad \left. \left. - \sum_{i < j \leq B} \mathcal{D}(y_i - y_j; \xi m^2) + \sum_{i < j \leq \bar{B}} \mathcal{D}(z_i - z_j; \xi m^2) \right) \right] \\
& \quad \times \langle A_\mu \dots \phi(y_1) \dots \phi(y_B) \phi^*(z_1) \dots \phi^*(z_{\bar{B}}) \rangle_0
\end{aligned} \tag{43}$$

where the correlation is evaluated with the lagrangian (37) in the  $R_\xi$  gauge. We observe that on the left-hand side, the source terms quadratic in  $\chi$  vanish in the limit  $m \rightarrow 0$ , but not the terms linear in  $\chi$ . Hence, even in the massless limit, the above formula does not give the  $\xi$  dependence of the correlation of elementary fields alone. In the limit  $m \rightarrow 0$ , we obtain the following  $\xi$  dependence after renormalization in the MS scheme:

$$\begin{aligned}
& \left\langle A''_{\mu, r} \dots \phi_r(y_1) \dots \phi_r^*(z_1) \dots \right. \\
& \quad \times \exp \left[ i \xi_r e_r^2 v_r \int_r \chi_r(r) \left( \sum_{1 \leq i \leq B} \Delta(y_i - r; 0) - \sum_{1 \leq i \leq \bar{B}} \Delta(z_i - r; 0) \right) \right] \right\rangle_{\xi_r} \\
& = e^{\frac{\xi_r e_r^2}{(4\pi)^2} \frac{1}{\epsilon} (B - \bar{B})^2} \times \Pi_{i, j} \left( \pi e^\gamma \mu^2 (y_i - z_j)^2 \right)^{-\frac{\xi_r e_r^2}{(4\pi)^2}} \\
& \quad \times \Pi_{i < j \leq B} \left( \pi e^\gamma \mu^2 (y_i - y_j)^2 \right)^{\frac{\xi_r e_r^2}{(4\pi)^2}} \times \Pi_{i < j \leq \bar{B}} \left( \pi e^\gamma \mu^2 (z_i - z_j)^2 \right)^{\frac{\xi_r e_r^2}{(4\pi)^2}} \\
& \quad \times \langle A_{\mu, r} \dots \phi_r(y_1) \dots \phi_r^*(z_1) \dots \rangle_0
\end{aligned} \tag{44}$$

Note that the IR divergence in the first factor on the right-hand side is due to the long-range source term coupled to  $\chi_r$ .<sup>7</sup>

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<sup>7</sup>In the  $R_\xi$  gauge there is no IR divergence in the correlations of elementary fields. Here the IR divergences arise due to the long range function  $\Delta(x; 0) = O(1/x^2)$  in the source term.

In this paper we have given two new proofs of the result of Landau and Khalatnikov on the exact gauge dependence of the correlation functions of the elementary fields for the abelian gauge theories in the covariant gauge. The trick of the Stückelberg field can be introduced also to the non-abelian gauge theories, but due to the coupling of the Stückelberg field to the gauge and FP ghost fields, no simple formulas can be obtained.

I thank the referee for an appropriate suggestion for revision. The first version of the paper was written during a visit to the particle theory group of Hokkaido University. I would like to thank Prof. Noboru Kawamoto and the other members of the group for hospitality. This work was supported in part by the Grant-In-Aid for Scientific Research from the Ministry of Education, Science, and Culture, Japan (#11640279).

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